"The repulsive force between two small spheres charged with the same sort of electricity is in the inverse ratio of the squares of the distances between the centers of the two spheres."

—Augustin Coulomb,
Memoir for 1785 of the French Academy of Sciences

Chapter 2

Coulomb's Law for Static Electricity, Principle of Superposition

Chapter Overview

Section 2.2 discusses the discovery of the inverse square law, and Section 2.3 presents the law explicitly. In Section 2.4, we estimate the characteristic force on an electron in an atom. Since the electrical force holds atoms, molecules, cells, and tissues together, from the size of this atomic force we can estimate the strength of materials. Section 2.5 introduces and applies the principle of superposition, which holds for the addition of forces of any origin, both electrical and nonelectrical. Section 2.6 shows two ways in which symmetry considerations can be used to simplify calculations. Section 2.7 considers the force on a point charge due to a charged rod, both using numerical integration (quadrature) and the analytical methods of integral calculus. Section 2.8 discusses problem solving and study strategy.

■

2.1 Introduction

The previous chapter summarized what was known, both qualitatively and quantitatively, up to about 1760. It included the law of conservation of charge. The present chapter presents another quantitative law: Coulomb's law for the force on one point charge due to another, which varies as the inverse second power of their separation—a so-called *inverse square* law. Chapters 1 through 6 all deal with what is called static electricity, as produced, for example, by rubbing a comb through your hair.

2.2 Discovering the Laws of Static Electricity

Optional

Before the spacial dependence of the electrical force had been established, the following results were already known:

1. There are two classes of electric charge; those in the same class repel, and those in different classes attract. This is summarized by the statement that "opposites attract and likes repel" (Dufay).

2. The electrical force lies along the line of centers between the two charges.

3. The force is proportional to the amount of charge on each of the objects. This proportionality seems to have been known by Aepinus, who, in 1759, knew everything about the force law except its specific dependence on distance.

There were also a number of observations that pointed, either directly or indirectly, to the electrical force satisfying an inverse square law.

Gray. Around 1731, Gray had found that two oak cubes, one solid and the other hollow, of the same exterior dimensions, received electricity in equal amounts when connected by a slightly conducting "pack thread" that was touched in the middle by an electrified tube. (If oak is not dry enough, it serves as an electrical conductor.) This fact implies that for both solid and hollow cubes, the charge resides on the outer surfaces. At the time no one realized that such surface charging implies an inverse square law for the electrical force.

Franklin and Priestley. Around 1755, Franklin noticed that an uncharged, insulated cork lowered into a charged metal cup is neither attracted to the cup's interior surface nor gains electricity on contacting that surface. He encouraged his friend Priestley to investigate this phenomenon further. The latter concluded, in 1767, by an analogy to gravity, that this implied the force law was an inverse square. Newton previously had showed that there is no gravitational force on a mass within a shell of uniform mass per unit area.

Robison. Robison performed electrical measurements using a balance that countered the torque due to electrical repulsion by the torque due to gravitation, finding in favor of an inverse square law. See Figure 2.1, where the plane of the holder assembly (including the charged spheres) is normal to the axis of the handle. Robison did not publish his results of 1769 until 1801, in a supplement to the *Encyclopedia Brittanica*.

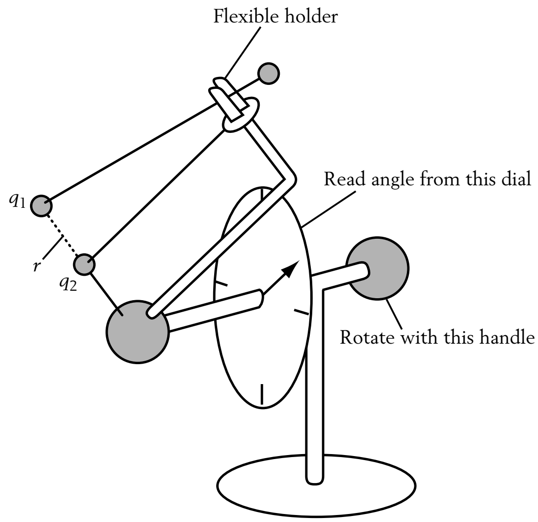

Figure 2.1 Robison's gravity balance. Spheres are given charges of the same sign, and they repel until the electrical repulsion balances the pull of the earth's gravity on the upper sphere. For a given orientation angle, the separation is measured.

Cavendish. Cavendish—wealthy, eccentric, and reclusive—performed many important scientific studies, including torsion balance studies of the gravitational force between two spheres, and what are probably the first measurements of the relative electrical conductivities of different materials, obtained by comparing the shocks he received on discharging a Leyden jar through different wires. Probably inspired by Priestley's work, and thus thinking in terms of an inverse square law, in 1770 Cavendish measured the charge on the inner of two concentric metallic shells connected by a fine wire. He found it to be zero, within experimental accuracy. This indirect method established that, if the electrical interaction satisfies a power law, then within a few percent it is an inverse square law. Not until Maxwell read Cavendish's notebooks, nearly 100 years later, was it appreciated how much Cavendish had done and understood.

Coulomb. Coulomb, a military engineer, performed numerous first-rate studies in physics, including friction and the elasticity of thin wires. The latter work led him to invent the torsion balance (independently of Cavendish), which he used to study static electricity and permanent magnets. See Figure 2.2. For springs, Hooke (around 1650) found that the force F opposing a length change by x is proportional to x: $F = -Kx$, where K is a measureable *spring constant*. Coulomb found a similar relation for torsion fibers: the torque τ opposing an angular twist by ϕ (in radians) is proportional to ϕ: $\tau = -\kappa\phi$, where κ is a measureable *torsion constant*. The torque could thus be determined from the angular displacement. Since the moment arm l was known, the force magnitude $|F|$ could then be deduced, via $|F| = |\tau/l| = |\kappa\phi/l|$.

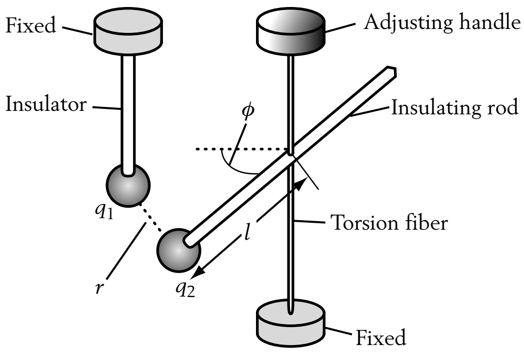

Figure 2.2 Coulomb's torsion balance. Spheres are given charges of either sign, and they rotate until the torque from the electrical force balances the torque from the torsion fiber. For a given orientation angle, the separation is measured.

Coulomb charged up two spheres equally (see Figure 1.8) and found that the force decreased with time. This he attributed to a loss of electric charge. He eliminated some of this decrease by improving the insulation in the supports. However, there was additional loss, due to the atmosphere, which was more extreme in humid weather. Accounting for the rate at which the charge decreased from his spheres improved the accuracy of his measurements. By using electrostatic induction, he produced oppositely charged spheres (see Figure 1.12). (However, in analyzing his results, he did not include the effects of electrostatic induction: for each sphere he considered the charge to be located at its center.) Coulomb published

Effect of Cosmic Rays

Even in dry weather such loss of charge occurs due to stray positive and negative ions in the air; the sphere attracts ions of charge opposite to its own. Only around 1910 was it discovered that such ions are produced by high-energy particles from outer space, called *cosmic rays*. Most cosmic rays are protons.

his work, which was well known in his native France, but 20 years passed before it was to be appreciated elsewhere.

2.3 The Inverse Square Law of Electricity: Coulomb's Law

Robison, Cavendish, and Coulomb all concluded that the electric force between two distinct point objects with charges q and Q varies as the inverse square of the separation. Thus

$$|\vec{F}| = \frac{k|qQ|}{r^2}, \qquad \text{(force between two charges)} \tag{2.1}$$

where the constant k depends upon the units for force, distance, and charge.

When charge is measured in SI units of the coulomb, distance is measured in terms of meters, and force in terms of newtons, the constant k can be determined. It takes on the value

$$k = 8.9875513 \times 10^9 \frac{\text{N-m}^2}{\text{C}^2}, \tag{2.2}$$

which usually will be taken to be $k = 9.0 \times 10^9$ N-m^2/C^2. In later chapters, it also will be useful to use the quantity ϵ_0, called the *permittivity constant*, or the *permittivity of free space*, given by

$$k = \frac{1}{4\pi\epsilon_0}, \qquad \epsilon_0 = \frac{1}{4\pi k} = 8.85418781762 \times 10^{-12} \frac{\text{C}^2}{\text{N-m}^2}. \tag{2.3}$$

SI units were not available to Coulomb, but that was not necessary in order to establish the inverse square law. Recall that the coulomb is defined in terms of the unit of electric current, which is the ampere. Thus a coulomb is the amount of charge that passes when an ampere of current flows for one second, or C = A-s.

The results (1–3) in Section 2.2, and (2.1), can together be expressed as a single vector equation for the force $\vec{F}$ on a charge q due to a charge Q. *When specifying a vector, we will employ an arrow or—if it is a unit vector—a hat above it.* Using the notation that the unit vector $\hat{r}$ points toward the observation charge q, from the source charge Q, we have

$$\vec{F} = \frac{kqQ}{r^2}\hat{r}. \qquad \text{(force } \vec{F} \text{ on charge } q,\ \hat{r} \text{ toward } q) \tag{2.4}$$

The bare geometry in the problem statement (i.e., what the problem provides) is given in Figure 2.3(a). No matter what the charges q and Q, in finding the force on q, $\hat{r}$ points to q from Q.

If q and Q are like charges, the force on q also points to q from Q. The corresponding geometry solving the problem (i.e., what the student must provide) is

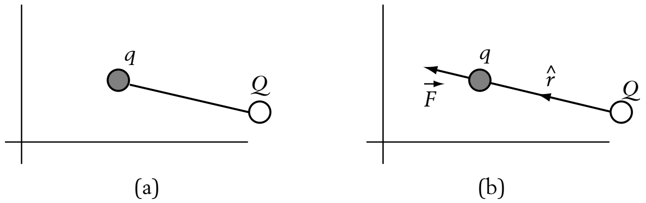

Figure 2.3 Geometry of interacting charges of the same sign. (a) The charges alone. (b) The geometry associated with the force $\vec{F}$ acting on q, with unit vector $\hat{r}$ pointing to q.

given in Figure 2.3(b). The tail of the force on q is placed on q. In this chapter, we will sometimes give both the bare geometry and the solution geometry that students must learn to provide; in later chapters, we will give only the latter.

To obtain the force on Q due to q, we use the unit vector to Q from q, which is opposite the unit vector to q from Q. Hence the force on Q due to q is opposite the force on q due to Q, and action and reaction is satisfied. A figure of the force on Q would place the tail of the force on Q.

2.4 Simple Applications of Coulomb's Law

As mentioned repeatedly, the electrical force, or C force, holds together atoms, molecules, and solids, and indeed holds together our very bodies. For that reason, it is important to get a feeling for how large a force it provides, both within atoms and within nuclei. We will not pursue these questions in great detail because classical mechanics (i.e., Newton's laws) cannot be applied *literally* at such small distances. In that case, *quantum mechanics*, an advanced topic, provides an accurate description.

2.4.1 *Coulomb's Law, Atoms, and the Strength of Materials*

A good rule of thumb is that atoms have a characteristic dimension of about 10^{-10} m, a unit that has been named the *angstrom*, or Å. Some atoms are larger, and some are smaller, but that is a good starting point. (Remember, it is more important to get the exponent correctly than to get the prefactor, although both are needed for precision work.) Therefore, consider the force on an electron in a hydrogen atom, using a separation of $r = 10^{-10}$ m. (Actually, for the hydrogen atom, the appropriate distance is about half that.) Coulomb's law, with both the electron and proton having the same magnitude $|q| = |Q| = e = 1.6 \times 10^{-19}$ C for the charge, yields for the electrical force between the electron and proton $F^{el}_{e,p} = ke^2/r^2 = 2.3 \times 10^{-8}$ N. This appears to be small, but not in comparison with the force on an atom due to the earth's gravity. Take $m_p = 1.67 \times 10^{-27}$ kg for the atomic mass (essentially, the mass of the proton since the electron is so much less massive). Then, with $g = 9.8$ m/s^2, $F^{grav}_{p,earth} = mg = 1.64 \times 10^{-26}$ N. Thus, comparison of the electrical force within the atom to the earth's gravitational force on the atom shows that the latter is negligibly small. This has profound structural significance for individual atoms and even for large molecules: their structure is indifferent to the local gravitational environment. Only on the scale of larger objects, such as trees and people, does gravity affect structure.

Application 2.1 **Electron–proton gravity within the atom is negligible**

The gravitational force of attraction between the electron and the proton is extraordinarily small. With $G = 6.67 \times 10^{-11}$ N-m^2/kg^2 and $m_e = 9.1 \times 10^{-31}$ kg, we obtain $F_{e,p}^{grav} = Gm_e m_p/r^2 = 1.01 \times 10^{-47}$ N, a force about 10^{39} times smaller than the electric force between them.

Application 2.2 **Material strength is atomic force per atomic area**

Let us take an interatomic force of 10^{-8} N to correspond to the force between nearby atoms in a bulk material. Taking atoms to be typically about 3×10^{-10} m apart, so with a cross-section of an atomic separation squared, or $(3 \times 10^{-10}\ \mathrm{m})^2 = 9 \times 10^{-20}\ \mathrm{m}^2 \approx 10^{-19}\ \mathrm{m}^2$, this gives a force per unit area of on the order of 10^{11} N/m^2. A commonly measured property of materials is the force per unit area needed to produce a given fractional change in the atomic separation. This is known as the *elastic constant*. For real materials, the elastic constants are also on the order of 10^{11} N/m^2. This agreement indicates (but does not prove) that electrical interactions are responsible for the elastic properties of materials. Assuming that breakage occurs when the fractional change in atomic separation is on the order of 0.1 gives a tensile stress on the order of 10^{10} N/m^2, much higher than for real materials: the tensile stress of iron is on the order of 10^9 N/m^2, and for string it is on the order of 10^7 N/m^2. This indicates that something else determines when a material breaks. In the 1930s, it was discovered that details of atomic positioning, and slippage at the atomic level via what are called *dislocations*, are responsible for the relatively low tensile stress of most materials.

Estimate of adhesive strength. Let us take a modest interatomic force of 10^{-10} N to correspond to what might occur for an adhesive. Let us also take there to be one such interatomic force per 10^{-7} m in each direction along the surface. (This corresponds to about 1 every 1000 atoms.) For a 1 cm^2 = 10^{-4} m^2 area, there are $10^{-4}\ \mathrm{m}^2/10^{-14}\ \mathrm{m}^2 = 10^{10}$ such interatomic forces, leading to a net force of 1 N, an appreciable value. Clearly, the Coulomb force is strong enough to explain the behavior of adhesives—and the adhesion between living cells.

2.4.2 *Nuclei and the Need for an Attractive Nuclear Force*

The Coulomb force also acts within atomic nucleii, whose characteristic dimension is 10^{-15} m, which is called a *fermi*. There are two protons in a He nucleus, which repel each other because of the Coulomb force. We could compute this force from (2.1), but it is easiest to obtain it by noting that in this case the charges have the same magnitude as for the electron and proton of the previous section, but the distances are smaller by a factor of about 10^5. Since the Coulomb force goes as the inverse square, the force of repulsion between two protons in a helium nucleus is larger by about 10^{10} relative to the electron–proton force in an atom. Thus $F^{el}_{p,p} = 10^{10} F^{el}_{e,p} = 2.3 \times 10^2$ N, which would support a mass exceeding 20 kg under the earth's gravity. This is enormous for such a small object. What keeps the nucleus from blowing apart is an attractive *nuclear force* between the nucleons (protons and neutrons). This force has a very short characteristic range, on the order of a fermi, but it is very strong so that within its range it can dominate the Coulomb repulsion.

Coulomb Repulsion and the Types of Helium Nuclei

Consider the possible types of He nuclei. Because of the Coulomb repulsion, too many protons relative to neutrons is bad for nuclear stability. That is why there is no such thing as a stable ^{2}He nucleus: only one pair of attractive nuclear interactions (between the two protons) is insufficient to overcome the Coulomb repulsion. On the other hand, ^{3}He has two protons and one neutron, with three pairs of attractive nuclear interactions (proton-proton, and two proton-neutron) that overcome the Coulomb repulsion (proton-proton). The isotope ^{4}He, with two protons and two neutrons, has six pairs of attractive nuclear interactions, and is yet more stable than ^{3}He. Additional neutrons must reside in nuclear orbitals that are far from the center of mass of the nucleus, and thus do not participate fully in the attractive interaction of the other nucleons. The isotopes ^{5}He and ^{6}He, although observed experimentally, are unstable, and helium nuclei with larger numbers of neutrons have not been observed at all. Neutron stars exist only because they are so massive that the gravitational attraction is large enough to keep them together.

2.4.3 *A Simple Charge Electrometer: Measuring the Charge Produced by Static Electricity*

Charge electroscopes (such as gold-leaf electroscopes, or the aluminum-foil electroscope of Figure 1.25) and *charge electrometers* are devices for measuring the charge on an object, the electrometer being more quantitative. They use the repulsive force between like charges. Figure 2.4 depicts an experiment to determine how much electricity can be produced by rubbing. Hanging from a common point are two threads of length l and two identical small conducting spheres of mass m, which have been given the same charge q by the charge-sharing process described in Section 1.5.2. Let us find the relationship between the angle θ and the charge q; clearly, the larger the charge, the larger the angle of separation.

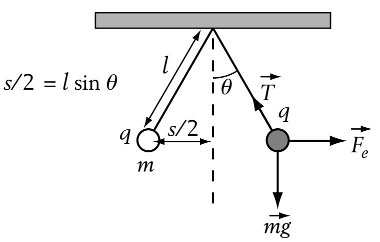

Figure 2.4 A simple electrometer. The two spheres are of equal mass m and equal charge q. By measuring the separation s or the angle θ (which are related), the electric force and the charge can be determined.

This is a problem in statics, where each ball has three forces (each with a different origin) acting on it: gravity (downward), electricity (horizontally, away from the other ball), and the string tension T (along the string, from the ball to the point where the string is taped to the stick). Neither the electrical force nor the tension is known. We first discuss some geometry: from the separation s we can deduce the angle that the strings make to the normal:

$$\sin\theta = \frac{s/2}{l} = \frac{s}{2l}. \tag{2.5}$$

From the equations of static equilibrium applied to either ball (to be explicit, we'll consider the ball on the right), we can obtain two conditions. These can be used to eliminate the tension and to relate the electrical force to the gravitational force and to the geometry of the problem.

First, the ball on the right (which we will consider to be the *observer*) feels a downward force of mg from gravity, and an upward force component $T\cos\theta$ from the string tension. In equilibrium, since the sum of the vertical forces is zero, or $\sum F_y = 0$, we have $mg = T\cos\theta$. This can be rewritten as

$$T = \frac{mg}{\cos\theta}. \tag{2.6}$$

In addition, the ball feels a rightward electrical force $F_e = kq^2/s^2$ and a leftward force component $T\sin\theta$ from the string tension. In equilibrium, the sum of the horizontal forces is zero, or $\sum F_x = 0$, so

$$F_e = \frac{kq^2}{s^2} = T\sin\theta = mg\tan\theta, \tag{2.7}$$

where we have eliminated T by using (2.6). Solving for q, we obtain

$$q = s\sqrt{\frac{mg\tan\theta}{k}}. \tag{2.8}$$

To be specific, take length $l = 10$ cm $= 0.1$ m, mass $m = 4$ g $= 0.004$ kg, and separation $s = 2.5$ cm $= 0.025$ m. Then, by (2.5), $\sin\theta = 0.125$, so $\cos\theta = \sqrt{1-(0.125)^2} = 0.992$ and $\tan\theta = 0.125/0.992 = 0.126$. Using m, s, and θ from the statement of the problem, and $g = 9.8$ m/s^2, (2.8) gives $q = 1.85 \times 10^{-8}$ C as a typical amount of charge that can be obtained by rubbing a comb through one's hair. By analyzing Figure 2.4 quantitatively, we have turned a qualitative electroscope into a quantitative electrometer!

At small θ, where $\sin\theta$ and $\tan\theta$ both vary as θ, s varies as θ and thus q varies as $\theta^{\frac{3}{2}}$. Correspondingly, θ varies as $q^{\frac{2}{3}}$. This rises very quickly for small q, so a measurement of angle is quite sensitive for very small charge. However, for very large charge, all angles will be near 90°, so a measurement of angle is insensitive for very large charge.

A complete solution would require the tension T of the string, given by (2.6). This quantity becomes relevant if we have a weak string that easily can be broken. In most mechanics problems, physicists don't worry about such questions, but mechanical and civil engineers make a living out of them.

This problem has gravity, strings, and electricity, and at first it seems like apples and oranges and bananas. Just as you can add the scalars representing the masses or calorie contents of different types of fruit, so you can add the vectors representing different types of force.

2.5 Vectors and the Principle of Superposition

2.5.1 *What We Mean by a Vector: Its Properties under Rotation*

Sections R.9 and R.10 discuss vectors in detail. If you aren't yet comfortable with vectors, and you haven't already read those sections, read them now.

It is so important to drive this message home that we'll repeat what you already know. Vectors are characterized by magnitude and direction—*and by their properties under rotation*. Quantities like force, position, velocity, and acceleration are vectors. Their magnitudes do not change under rotation, and the orientation between two such quantities does not change under rotation. If a position vector and a force vector are at 40° to each other, then after any rotation they remain at 40° to each other.

In contrast, consider pressure P, temperature T, and energy E. None of these three quantities change under a rotation in space; they are scalars. Hence the three-component object (T, P, E) does not transform as would a vector under rotations in space. Merely having three components doesn't assure "vectorness."

2.5.2 *The Principle of Superposition: Add 'Em Up*

Because forces are vectors, when there are individual forces acting on a single object, the net force is obtained by performing vector addition on all the forces. This is called the *principle of superposition*. We used this principle in the electrometer example, where the three forces each had a different source. In what follows, we will use the principle of superposition to add up many forces of electrical origin. A force $\vec{F}$ may have components along the x-, y-, and z-directions. We specify these directions as the set of unit vectors $(\hat{x}, \hat{y}, \hat{z})$, or $(\hat{i}, \hat{j}, \hat{k})$. Indeed, we will sometimes write $\hat{\rightarrow}$ for $\hat{i}$ and $\hat{\leftarrow}$ for $-\hat{i}$, $\hat{\uparrow}$ for $\hat{j}$ and $\hat{\downarrow}$ for $-\hat{j}$, and $\hat{\odot}$ for $\hat{k}$ (out of page) and $\hat{\otimes}$ for $-\hat{k}$ (into page). Thus $(\hat{x}, \hat{y}, \hat{z}) \equiv (\hat{i}, \hat{j}, \hat{k}) \equiv (\hat{\rightarrow}, \hat{\uparrow}, \hat{\odot})$. You should already be accustomed to seeing numerous ways to write the same thing. For certain problems involving gravity, it is often convenient to let *down* correspond to $\hat{y}$. The ways in which we write the laws of physics all depend upon conventions (e.g., we use right-handed, not left-handed, coordinate systems). However, the laws themselves do not depend on these conventions.

Explicitly, we write

$$\vec{F} = F_x\hat{x} + F_y\hat{y} + F_z\hat{z}, \tag{2.9}$$

where F_x is the x-component of $\vec{F}$, and so forth. By the rules of the scalar product, where $\hat{x} \cdot \hat{x} = 1$, $\hat{x} \cdot \hat{y} = 0$, and so forth, we have

$$\vec{F} \cdot \hat{x} = F_x. \tag{2.10}$$

Similarly, we can determine F_y and F_z. Thus, if $\vec{F}$ is specified in terms of its magnitude $F = |\vec{F}|$ and its direction, it also can be written in terms of its components.

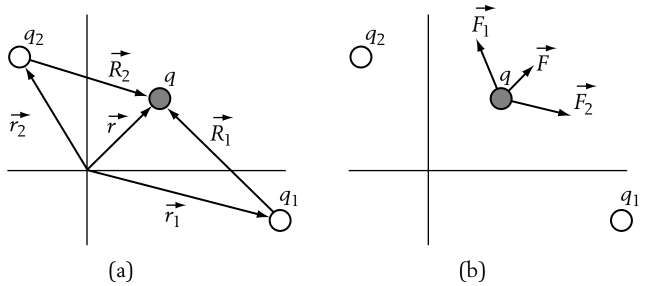

Figure 2.5 (a) Geometry for the force on q due to q_1 and q_2. Lowercase vectors refer to distances from the origin, and uppercase vectors refer to relative distances.
(b) Force on q, due to q_1 and q_2.

Likewise, if $\vec{F}$ is specified in terms of its components, then its magnitude can be computed from

$$|\vec{F}| = \sqrt{F_x^2 + F_y^2 + F_z^2}. \tag{2.11}$$

When, in addition to the charge q at $\vec{r}$, there are many other charges, by the principle of superposition the total force is the vector sum of all the forces acting on q. Consider a situation with many charges q_i at respective positions $\vec{r}_i$. Figure 2.5(a) depicts only q_1 and q_2, in addition to q. For the pair q and q_1, we have $\vec{r} = \vec{R}_1 + \vec{r}_1$, so $\vec{R}_1 = \vec{r} - \vec{r}_1$. More generally, $\vec{R}_i = \vec{r} - \vec{r}_i$. With $\hat{R}_i = \vec{R}_i/|\vec{R}_i|$, we see that $\hat{R}_i$ *points to the observation charge* q at $\vec{r}$. For example, if in Figure 2.5(a) $\vec{r} \equiv (4, 4, 0)$ and $\vec{r}_1 \equiv (16, -3, 0)$, then $\vec{R}_1 \equiv (-12, 7, 0)$, $|\vec{R}_1| = \sqrt{193} = 13.89$, and $\hat{R}_1 \equiv (-0.864, 0.504, 0)$.

We generalize (2.4) to obtain the force on q due to q_i as $\vec{F}_i = (kqq_i/R_i^2)\hat{R}_i$. Summing over all $\vec{F}_i$ yields the total force on q. Explicitly, it is

$$\vec{F} = \sum_i \vec{F}_i = \sum_i \frac{kqq_i}{R_i^2}\hat{R}_i.$$

$$\text{(force } \vec{F} \text{ on charge } q \text{ at } \vec{r},\ \hat{R}_i \text{ toward } q,\ \ \vec{R}_i = \vec{r} - \vec{r}_i) \tag{2.12}$$

Let's talk in terms of input and output. The input consists of the charge q and its position $\vec{r}$, and the charges q_i at the positions $\vec{r}_i$. For two source charges, this is given in Figure 2.5(a). The output consists of the individual forces $\vec{F}_i$ and their vector sum $\vec{F}$. This is depicted in Figure 2.5(b) for the cases where the charges q, q_1, and q_2 are all positive or all negative. The relative lengths of $\vec{F}_1$ and $\vec{F}_2$ can only be determined when actual values for q, q_1, and q_2 are given; therefore, Figure 2.5(b) is only a schematic. Each force on q is drawn with its tails on q, as if a person were pulling on a string attached to q.

Example 2.1 Adding force magnitudes almost always produces garbage

Adding vector magnitudes to obtain the magnitude of a sum of vectors only works when the vectors have the same direction. Thus, adding vector magnitudes usually gives incorrect answers. For example, if two horses pull on opposite sides of a rope, each with 200 N force, the resultant vector force on the rope is zero, so its magnitude is zero, not 400 N.

Let's dignify this important result with the unnumbered equation

$$\vec{F} = \vec{F}_1 + \vec{F}_2, \text{ but } |\vec{F}| \le |\vec{F}_1| + |\vec{F}_2|, \text{ when adding two forces.}$$

Another example of this result is given by the forces in Figure 2.5(b). This sort of equation (called a constraint) holds when we add two vectors of any type.

Example 2.2 Cancellation of two collinear forces

Consider that, for two charges q_1 and q_2 whose positions are known, we would like to know where to place a third charge q so that it feels no net force due to q_1 and q_2. We are free to choose a geometry where the charges are along the x-axis, with q_1 at the origin, and q_2 a distance l to its right. Thus our specific question is: where should q be placed in order to feel no net force? (Our answer will be independent of q, a fact that is related to the concept of the electric field, which will be introduced in the next chapter.) Before performing any calculations, note that *for there to be zero net force, the two forces on q must have equal magnitude but opposite direction*. The latter condition can only hold if the third charge is placed on the line determined by q_1 and q_2. There are two possibilities.

(a) **q_1 and q_2 have the same sign.** In this case, q should be placed between q_1 and q_2, at some distance s from q_1 (to be determined) where q will feel canceling attractions (or repulsions) $\vec{F}_1 = -\vec{F}_2$ from q_1 and q_2. To be specific, let all charges be positive. See Figure 2.6(a). Then $\vec{F}_1$ points away from q_1 and $\vec{F}_2$ points away from q_2, by "likes repel." As in Figure 2.5(b), the forces on q are drawn with their tails on q. Note that the position s where the forces $\vec{F}_1$ and $\vec{F}_2$ cancel will not change if the signs of all the charges are reversed, because again "likes repel." Moreover, the position s where the forces $\vec{F}_1$ and $\vec{F}_2$ cancel will not change if the sign of q is reversed, or if the signs of both q_1 and q_2 are reversed, because then, although the forces change direction, they still cancel.

Let the magnitudes $F_{1,2} = |\vec{F}_{1,2}|$. By (2.1) or (2.4), the equilibrium condition that $F_1 = F_2$ gives

$$\frac{kqq_1}{s^2} = \frac{kqq_2}{(l-s)^2}. \tag{2.13}$$

Canceling kq, taking the positive square root (remember, we have already

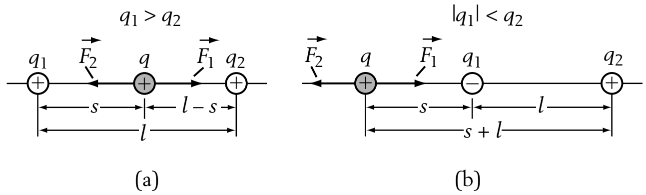

Figure 2.6 Locating the zero-force position. (a) When the two source charges have the same sign. (b) When the two source charges have opposite sign. How would this figure look for $q < 0$?

determined that $0 < s < l$), and inverting each side, we obtain

$$\frac{s}{\sqrt{q_1}} = \frac{l-s}{\sqrt{q_2}}. \tag{2.14}$$

Solving for s, we obtain

$$s = \frac{l}{\sqrt{q_2/q_1}+1}. \tag{2.15}$$

This value of s is independent of the value of q; doubling q doubles each force, so they continue to cancel at the same position. As checks, note that (1) $s = l/2$ for $q_1 = q_2$ (i.e., the test charge is equidistant between two equal charges); (2) $s/l \to 0$ as $q_2/q_1 \to \infty$ (i.e., the test charge approaches the weaker charge, here q_1); and (3) $s/l \to 1$ as $q_2/q_1 \to 0$ (i.e., again the test charge approaches the weaker charge, now q_2).

Dimensionless Ratios

In taking limits where some quantity goes to zero or to infinity, it is often convenient to make that quantity a dimensionless ratio. For the test charge to approach the weaker charge, l is fixed and s gets smaller. We could obtain the same limit by keeping s fixed and letting l get larger. Using the dimensionless ratio s/l takes care of both possibilities.

(b) **q_1 and q_2 have opposite sign.** It will be sufficient to consider the case $q_1(<0)$, $q_2(>0)$, and $q(>0)$, because the other cases of this type are related to this one simply by changes in direction of both forces. To be specific, let $|q_1| < |q_2|$. Then q should be placed on the same line, but to the left of the weaker charge q_1, so that proximity can compensate for weakness. See Figure 2.6(b). The two forces will cancel when q is placed a distance s (to be determined) to the left of q_1. By (2.1), the equilibrium condition that $F_1 = F_2$ gives

$$\frac{kq|q_1|}{s^2} = \frac{kq|q_2|}{(l+s)^2}. \tag{2.16}$$

Solving for s as before, we now obtain

$$s = \frac{l}{\sqrt{|q_2/q_1|}-1}, \qquad (|q_2/q_1| > 1). \tag{2.17}$$

As checks, note that (1) $s \to \infty$ as $|q_2/q_1| \to 1$ (i.e., for equal strength source charges, the only way their forces cancel out, if one is nearer q than the other, is if q is at infinity); and (2) $s \to 0$ as $|q_1/q_2| \to 0$ (i.e., the test charge approaches the weaker charge, here q_1).

One Size Doesn't Always Fit All

We are often tempted to develop a totally general equation that will include all cases—"one size fits all." However, if we generalize too soon, we might not notice some fundamental distinctions. This often occurs in computer programming, where an algorithm developed for one situation does not work properly when applied to another. The related cases of like and unlike charges really should be treated separately.

Example 2.3 Addition of two noncollinear forces

Now consider the force on q due to q_1 and q_2 when the three charges do not lie along a line. In principle, the charges can be anywhere with respect to a fixed coordinate system, each of them requiring three numbers (x, y, z) to specify its position. However, because there are only three charges, and three noncollinear points define a plane, which we may choose to be the xy-plane, only two numbers (x, y) per charge will be required to specify each position. Further, we can place q at the origin, so $\vec{r} = \vec{0}$, leaving only two numbers (x, y) per charge to specify the positions of q_1 and q_2. The last allowed simplification is to take q_1 to lie along a specific direction, such as $\hat{x}$. The third charge, q_3, can lie anywhere in the xy-plane. Thus, the geometry is specified with a total of three numbers, which we take to be the distances R_1 and R_2 of q_1 and q_2 to the origin, and the angle θ_2 that R_2 makes with respect to the x-axis (we have already taken $\theta_1 = 0$). The problem is to find the net force on q. To be specific, let $q = 2.0 \times 10^{-9}$ C, $q_1 = -4.0 \times 10^{-9}$ C and $q_2 = 6.0 \times 10^{-9}$ C, $R_1 = 0.2$ m, $R_2 = 0.3$ m, and $\theta_2 = 55°$. See Figure 2.7(a).

Find the net force on q.

Solution: There are at least two ways in which we can proceed to solve this problem. We shall call one the *common sense* method, which is particularly appropriate when there are only a few forces involved. (We call it the common sense method because it has been said, with much truth, that science is simply common sense, but more refined.) We shall call the other method the *formal* method because it is a bit akin to the higher mathematics to which J. J. Thomson was referring in the quote at the end of Section R.5. When we have a choice, the first method is preferable, but there are times when the only practical method is the second one. We will use both in the present case.

If you don't know at least two ways to solve a problem, you may not really understand the problem in a deeper sense. Competent practitioners in any area can solve a given problem in multiple ways.

The common sense method first finds the magnitude of each force acting on q (the observer) by using (2.1), then gets the direction of each force by "opposites attract, likes repel," and finally performs the vector addition. That is more or less what we did in the previous example. The formal method uses (2.12) to compute each force in terms of its vector components (so we actually compute the magnitudes of the individual forces), and adds up the components.

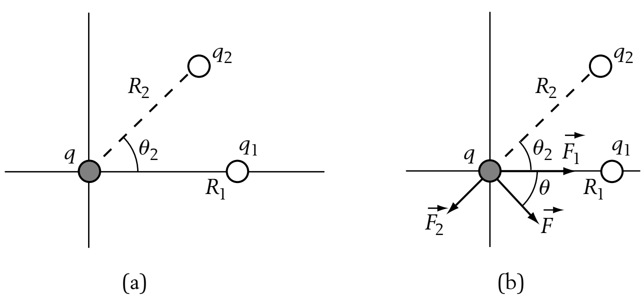

Figure 2.7 Force on q at origin, due to q_1 and q_2.
(a) Geometry of the problem. (b) Solution of the problem in terms of individual forces and the total force.

Common sense method. First, we find the magnitude $F_1 = |\vec{F}_1|$ of the force acting on q due to q_1. By Coulomb's law, as in (2.1),

$$F_1 = \frac{k|qq_1|}{R_1^2} = \frac{(9 \times 10^9 \text{ N-m}^2 / \text{C}^2)|(2 \times 10^{-9} \text{ C})(-4 \times 10^{-9} \text{ C})|}{(0.2 \text{ m})^2}$$
$$= 1.8 \times 10^{-6} \text{ N}.$$

Note that F_1 must be positive because it is the magnitude, or absolute value, of $\vec{F}_1$. Similarly,

$$F_2 = \frac{k|qq_2|}{R_2^2} = \frac{(9 \times 10^9 \text{ N-m}^2 / \text{C}^2)|(2 \times 10^{-9} \text{ C})(6 \times 10^{-9} \text{ C})|}{(0.3 \text{ m})^2}$$
$$= 1.2 \times 10^{-6} \text{ N}.$$

Using "opposites attract, likes repel," we can now draw the force diagram (compare Figure 2.7b). This leads to the force components

$$F_x = F_1 - F_2 \cos\theta_2 = 1.8 \times 10^{-6} \text{ N} - 1.2 \times 10^{-6} \text{ N} \cos 55^\circ$$
$$= 1.112 \times 10^{-6} \text{ N},$$
$$F_y = -F_2 \sin\theta_2 = -0.983 \times 10^{-6} \text{ N}.$$

Thus, as in Figure 2.7, $\vec{F}$ lies in the fourth quadrant. We also have

$$F = |\vec{F}| = \sqrt{F_x^2 + F_y^2} = 1.484 \times 10^{-6} \text{ N},$$

and $\vec{F}$ makes an angle with slope

$$\tan\theta = \frac{F_y}{F_x} = -0.884,$$

corresponding to an angle θ in the fourth quadrant, with $\theta = -41.5^\circ$, or -0.724 radians. If $\vec{F}$ had been in the second quadrant, its tangent would also have been negative, so to obtain the correct angle we would have had to add 180° (or π radians) to the inverse tangent of $\tan\theta$.

Finding $\vec{F}$ for Rotated Source Charges

If the positions of both q_1 and q_2 are rotated by 24° clockwise, $\vec{F}$ too is rotated by 24° clockwise, to an angle of $-41.5 - 24 = -65.5^\circ$ relative to the x-axis. Its magnitude is unchanged. Thus, the value of F_x for the rotated charges would be $|\vec{F}|\cos(-65.5) = 0.613 \times 10^{-6}$ N. This way to calculate the rotated F_x is simpler than recomputing and adding the x-components of the rotated versions of the individual forces.

Formal method. Even with the formal method, there are at least two ways to proceed. We may rewrite (2.12) as

$$\vec{F} = \sum_i F_i^* \hat{R}_i, \qquad F_i^* \equiv \frac{kqq_i}{R_i^2}, \tag{2.12$'$}$$

or as

$$\vec{F} = \sum_i \frac{kqq_i}{R_i^3} \vec{R}_i, \qquad \vec{R}_i = \vec{r} - \vec{r}_i. \tag{2.12$''$}$$

Using (2.12′), we must first compute the *signed* force F_i^* due to the i charge, and then the unit vector $\hat{R}_i$. Thus, as an intermediate step, we can readily

Table 2.1 Force Calculation

1	2	3	4	5	6	7	8	9	10	11
x_1	y_1	z_1	X_1	Y_1	Z_1	R_1	F_1^*	$F_1^*\left(\frac{X_1}{R_1}\right)$	$F_1^*\left(\frac{Y_1}{R_1}\right)$	$F_1^*\left(\frac{Z_1}{R_1}\right)$

determine the magnitude of each of the individual forces, via $F_1 = |\vec{F}_i| = |F_i^*|$, and we can also readily determine the direction of the force. Use of (2.12′), although part of the formal approach, is closely related to the commonsense method. In Appendix B we use (2.12′) to solve the last example using a spreadsheet.

Equation (2.12″) requires fewer operations than (2.12′). This is because (2.12″) does not require the intermediate computation of $\hat{R}_i = \vec{R}_i/|\vec{R}_i|$. However, with (2.12″) we do not obtain the magnitude and direction of each individual force. Both procedures work. (Different strokes for different folks.) Table 2.1 indicates specifically what computations would have to be made using (2.12″) for the force $\vec{F}_1$ on q at (x, y, z) due to q_1 at (x_1, y_1, z_1). The table uses $X_1 = x - x_1$, and so on, and F_i^* of (2.12′). Columns 9, 10, and 11 contain the x-, y-, and z-components of $\vec{F}_1$.

2.6 Use of Symmetry

When the source charge is symmetrically placed, and the observation charge is at a position where this symmetry is evident, certain simplifications can be made. We will discuss two ways to use symmetry. First, in doing a computation we notice that certain terms must add or cancel. Second, by some general principle or principles, and the fact that force is a vector (so that it rotates when the source charges rotate), we learn that certain possibilities are disallowed.

Example 2.4 **Force due to two equal charges**

Let there be two equal source charges Q symmetrically placed on the x-axis, and the observation charge q be along the y-axis. In what direction does the net force on q point?

Solution: See Figure 2.8(a), where the individual forces $\vec{F}_1$ and $\vec{F}_2$, and their resultant $\vec{F}$ have been drawn. By symmetry, $\vec{F}$ must be along the y-axis because the x-components of the individual forces cancel.

Example 2.5 **Force due to two equal and opposite charges**

In Example 2.4, let one source charge be replaced by $-Q$, as in Figure 2.8(b). In what direction does the net force on q point?

Solution: By symmetry, the net force must be along the x-axis because now the y-components of the individual forces cancel.

The above arguments are computational in nature. Here is a noncomputational symmetry argument. Consider first the two equal source charges Q in Figure 2.8(a). Rotating them by 180° about the y-axis gives an equivalent

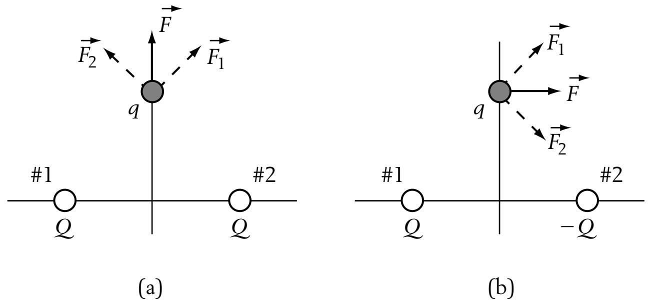

Figure 2.8 Force on q along perpendicular bisector between two charges. (a) Equal charges Q. (b) Equal and opposite charges $\pm Q$.

configuration, and therefore the net force on q does not change. Specifically, F_x does not change under this interchange of charges. However, for any vector (including the net force), a rotation about y must change the sign of its x-component. The only way for F_x to both change sign and not change sign is if $F_x = 0$. Consider now the equal and opposite source charges $\pm Q$ of Figure 2.8(b). Rotating by 180° about the y-axis, which interchanges the charges, must preserve the y-components of the forces. But interchanging the charges reverses the directions of the individual forces, including the y-components. The only way for F_y to both change sign and not change sign is if $F_y = 0$.

Example 2.6 **Force due to three equal charges**

Let there be three equal source charges $q_1 = q_2 = q_3 = Q$ placed at the corners of an equilateral triangle. Let an observation charge q be placed at the very center of the triangle. What is the net force on the observation charge, q?

Solution: See Figure 2.9, where the individual forces on q are given. The net force on q is zero, as can be established by algebraic computation, by numerical computation using particular values for the side of the triangle and the charges, and by the following argument. The forces on q due to each of these charges all have the same magnitudes, but are rotated by $\pm 120°$ relative to one another. The forces thus form the arms of an equilateral triangle that, under vector addition, sum to zero.

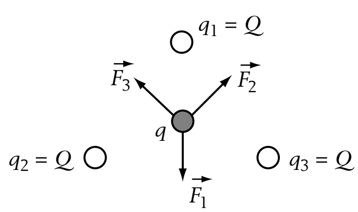

Figure 2.9 Individual forces on q at the center of three equal charges $q_1 = q_2 = q_3 = Q$ at the vertices of an equilateral triangle.

A noncomputational symmetry argument for Example 2.6 begins by observing that the net force must lie in the xy-plane. Because force is a vector, if the source charges are rotated by 120°, either clockwise or counterclockwise, the force must rotate in the same way. However, since $q_1 = q_2 = q_3 = Q$, this rotation produces the original charge distribution, and therefore the same force. Hence the force must be zero. For a large number (e.g., 47) of equal source charges at equal radii and angles, you can generalize this argument to show that the force on a charge at their center is also zero. For this more complex case, there is no computational symmetry argument. (Of course, for an even number of charges, the net cancellation is obvious because opposite pairs would produce canceling forces.)

2.7 Force Due to a Line Charge: Approximate and Integral Calculus Solutions

Let us obtain the force $\vec{F}$ on a charge q at the origin, due to a net charge Q that is uniformly distributed over the line segment from $(a, -l/2, 0)$ to $(a, l/2, 0)$. See Figure 2.10(a).

Note that $F_y = F_z = 0$, by symmetry. In principle, to find F_x requires calculus, where we break up the continuous line charge into an infinite number of infinitesimal point charges dQ, and add up their (vector) forces $d\vec{F}$ on q. See Figure 2.10(b). However, we will first consider what happens when we approximate the line charge by a finite number of point charges, and we add up their vector forces, using a spreadsheet.

Approximate approach. Spreadsheets can calculate numbers from algebraic formulas, but cannot perform algebra. (See Appendix B for an introduction to spreadsheets, in case you are not already familiar with them.) Therefore we will have to use specific values for q, Q, a, and l. We can employ our earlier problem, merely extending the number of rows to accommodate the number of charges in our approximation. Let us take $q = 10^{-9}$ C, $Q = 5 \times 10^{-9}$ C, $a = 1$ m, and $l = 7$ m. Now consider various approximations to the line charge Q.

1. Approximating Q by a single charge may be done by putting all of Q at the midpoint $(1, 0)$ of the line. In this case, there is no y-component, so all the force is along the x-direction. This gives a force of 45×10^{-9} N.
2. Approximating Q by two subcharges may be done by breaking the line into two equal segments of length $7/2$, and placing $Q/2$ at the segment midpoints, given by $(1, 7/4)$ and $(1, -7/4)$. This gives a force of 5.5×10^{-9} N.

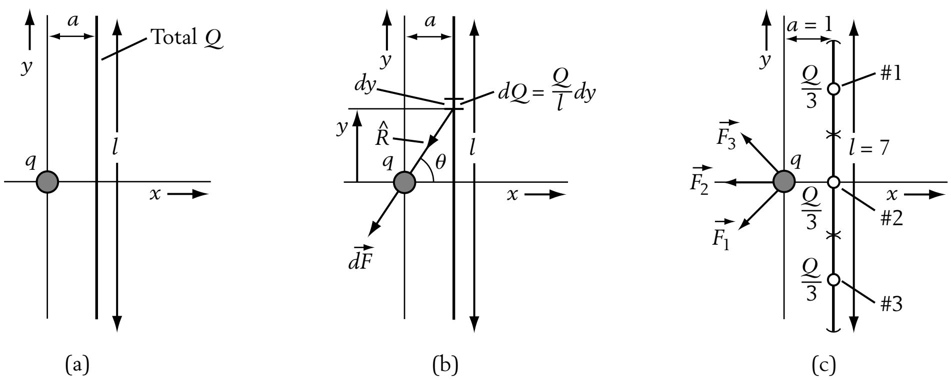

Figure 2.10 The force on a point charge due to a line charge. (a) Statement of the problem. (b) Force due to an element dq. (c) Force due to a discretization of the line into three equal charges.

Table 2.2 Force as a function of the number n of subcharges

n	1	2	3	4	5	6	7	8	9	10	11	12
F_n	45	5.50	16.83	10.60	13.22	12.01	12.53	12.30	12.40	12.3540	12.3714	12.3631

3. Approximating Q by three subcharges may be done by breaking the line into three equal segments of length 7/3, and placing $Q/3$ at the segment midpoints, given by $(1, 7/2 - (1/2)\cdot 7/3) = (1, 7/3)$, $(1, 7/2 - (3/2)\cdot 7/3) = (1, 0)$, and $(1, 7/2 - (5/2)\cdot 7/3) = (1, -7/3)$. See Figure 2.10(c). This gives a force of 16.83×10^{-9} N. [These calculations are worth doing yourself, either with pencil and paper or with a spreadsheet, to verify that you really understand how to use (2.12).]

More generally, we have Q/n on n subcharges, at the positions $(1, 7/2 - (1/2)\cdot 7/n)$, $(1, 7/2 - (3/2)\cdot 7/n)$, and so on. By using the calculation capability of the spreadsheet, we can compute F_n for various n. By the symmetry of the problem, only the x-components of the individual forces need be calculated. Moreover, the net force $\vec{F}$ points to the left, so F_x has a negative sign. Table 2.2 shows the *magnitude* of the sum of the x-components for n segments. It is given as F_n, and is in units of 10^{-9} N for n up to 12. The sums are converging.

Calculus approach. First set up the problem, which means picking out a typical piece of charge. Figure 2.10(b) shows one dQ in the interval dy centered at y. The line is divided into infinitely many dQ's spanning the range from $y = -l/2$ to $y = l/2$. Corresponding to this y is an angle that we call θ, and a direction $\hat{R}$ to the observation charge q. The dQ acts with a force $d\vec{F}$ on q. Our goal is to add up the $d\vec{F}$'s for all the dQ's that constitute the line charge, to obtain the total force $\vec{F}$ acting on q.

There is an order in which you must perform ordinary mathematical operations: multiplication and division are performed before addition and subtraction. Similarly, there is an order in which you must perform the mathematical operations associated with integral calculus: *identify the type of differential you are adding up; find its vector components (if it is a vector); then separately add up each set of components.* In the present case, we add up the x-components dF_x to obtain F_x, and so on.

As in the previous subsection, to find the force on q we need only consider the x-component of the force, or F_x. (By the symmetry of the problem, $F_y = 0$.) Now, instead of n segments of length $7/n$ and charge $(Q/7)(7/n) = (Q/n)$, we have an infinite number of segments of infinitesimal length $ds = dy > 0$. The charge per unit length is $\lambda = dQ/ds = (Q/l)$ because the charge Q is distributed uniformly over l. Thus dy has charge $dQ = \lambda dy = (Q/l)dy$. (Of course, if we add up all the dQ's, we will obtain Q.) We now apply the commonsense method to find dF_x due to dQ. For generality, instead of $a = 1$ and $l = 7$, we will employ the symbols a and l.

Notational Choices

There are many notational choices we can employ to solve a problem, especially in choosing intermediate variables. However, the final answer must be independent of intermediate notation.

From (2.1), the magnitude $dF = |d\vec{F}|$ of the force $d\vec{F}$ between q and dQ, separated by $R = \sqrt{a^2 + y^2}$ is $dF = kq(dQ)/R^2$. We now find the component dF_x of the force $d\vec{F}$ along x. From $d\vec{F}$ in Figure 2.10(b), this component is

$dF_x = -dF\cos\theta$. Then, with $dQ = (Q/l)dy$, we obtain

$$dF_x = -dF\cos\theta = -\frac{kq(dQ)\cos\theta}{(a^2+y^2)} = -\frac{kqQ\cos\theta dy}{l(a^2+y^2)}. \tag{2.18}$$

We've got to make a decision now: to eliminate y in favor of θ, or vice versa. We choose to eliminate y. Figure 2.10(b) shows that $y = a\tan\theta$. Thus

$$dy = (dy/d\theta)d\theta = a(d\tan\theta/d\theta)d\theta = a\sec^2\theta d\theta$$

and

$$a^2 + y^2 = a^2(1+\tan^2\theta) = a^2\sec^2\theta,$$

so (2.18) becomes

$$dF_x = -\frac{kqQ\cos\theta(a\sec^2\theta d\theta)}{la^3\sec^2\theta} = -\frac{kqQ\cos\theta}{al}d\theta. \tag{2.19}$$

We're now ready to actually employ the integral calculus (so far we've only used differential calculus to express dF_x). Before doing so, let's note that for two reasons this problem is more complex than integrating to add up the total charge Q on an object (as in Section 1.9). First, force is a vector, whereas charge is a scalar, so we have to determine vector components in this case. Second, the value of the force on q depends on its position, whereas the amount of charge on an object doesn't depend on the position of whoever is adding up that charge. In Chapter 5, where we discuss electrical potential, we will perform integrals of intermediate complexity: they are scalars (as for charge), but they depend on the position of the observer (as for force).

Now we do the integral to obtain F_x. We have

$$F_x = \int dF_x = -\frac{kqQ}{al}\int_{\theta_-}^{\theta_+}\cos\theta d\theta = -\frac{kqQ}{al}\sin\theta\bigg|_{\theta_-}^{\theta_+}. \tag{2.20}$$

From Figure 2.10(b), with $a = 1$ and $l = 7$, the maximum and minimum angles θ_+ and θ_- satisfy $\sin\theta_+ = (l/2)/\sqrt{a^2+(l/2)^2} = -\sin\theta_-$. Thus

$$F_x = -\frac{kqQ}{a\sqrt{a^2+(l/2)^2}}. \tag{2.21}$$

For $a = 1$ m, $l = 7$ m, $q = 10^{-9}$ C, and $Q = 5\times10^{-9}$ C, (2.21) yields $F_x = -12.362450\times10^{-9}$ N. It is very close to the value obtained numerically from Table 2.2. For a different set of inputs—$a = 3$ m, $l = 4$ m, $q = 4\times10^{-9}$ C, and $Q = -2\times10^{-9}$ C—(2.21) yields $F_x = -6.656\times10^{-9}$ N. The advantage of the general analytical expression (2.21) over brute force summation should be apparent. On the other hand, a computer can reevaluate a spreadsheet sum very quickly when the inputs change.

Checks are very important, in both numerical and analytical work. If you derive a result that is very general, it should work for specific cases whose answer you already know, and therefore you should look for specific cases that can serve

Look Again!

Because of what it will teach you about calculus and vectors, the derivation of (2.21) is in some ways the most important example in the book. Reread the statement of the problem and its solution until you understand it well enough to explain every step to someone else. Then try to reproduce the solution on your own. Understand it, don't memorize it. Learning how to solve this type of problem, involving integral calculus, represents a major level of intellectual achievement.

as checks. Without an analytic check that the spreadsheet or computer gives the correct result in at least one case, we cannot be sure that it gives the correct result in *any* case. With this in mind, comparison of our analytic work of (2.21) with Table 2.2 produced from the spreadsheet shows excellent agreement. Another check that can be made is to take the limit where the length l goes to zero; the line charge contracts to a point charge. Then (2.21) yields $F_x = -kqQ/a^2$, which is what we expect, by (2.1).

For an infinite wire, it is convenient to employ the charge per unit length λ, rather than the charge Q, which becomes infinite. In the present case, we have $\lambda = Q/l$, and now we let $l/a \to \infty$. Thus we can neglect a in $\sqrt{a^2 + (l/2)^2}$. Thus $Q/\sqrt{a^2 + (l/2)^2} \to Q/(l/2) = (2Q/l) = 2\lambda$. Hence (2.21) yields

$$F_x = -\frac{2kq\lambda}{r}, \qquad \lambda = Q/l, \qquad l/a \to \infty. \tag{2.22}$$

Here is a way to check this r^{-1} result. Consider two identical infinite, parallel rods, and an observation charge q in their plane at a position that makes one of the rods twice as far away. See Figure 2.11. Let a pair of closely spaced lines originate radially at the observer and intersect both rods. The distance r' to the intersection of the radial lines is twice as big for the further rod as for the nearer rod. If the charge intersected by the radial lines is dQ for the nearer rod, then the charge intersected is $2dQ$ for the further rod. By (2.4), with its inverse square dependence on distance and its linear dependence on charge, for each corresponding element intersected by the radial lines the force due to the further rod is $(1/2)^2 2 = 1/2$ that due to the nearer rod. Hence, if both rods are infinite, adding up the effects for all elements of charge will give a total force due to the further rod that is one-half that due to the nearer rod. This is in agreement with the r^{-1} result of (2.22). Only by doing the integral, however, can we obtain the coefficient of proportionality.

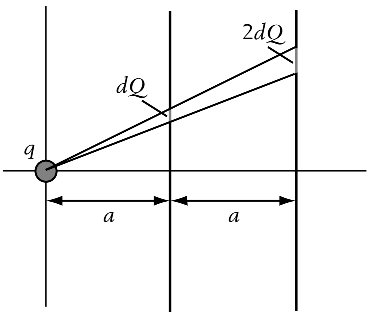

Figure 2.11 Charge q and elements of charge dq and $2dq$ from two rods of equal charge density. The magnitudes of the forces on q are the same in each case.

In contrast to the spreadsheet result, the integral calculus result is exact. If we set up the spreadsheet calculation with an arbitrary position for q, so we also compute F_y, then by changing the position of q, the spreadsheet will nearly instantly calculate the force at *any* position. However, numerous changes have to be made to obtain the integral calculus result. Although this problem—to find the force on q if it is placed at *any* position in the plane—can be done in a closed form, there are many problems where even a slight change in the observation point will cause the resulting integrals to be vastly more complicated or even unsolvable. It is straightforward to use calculus to obtain the force on a charge q placed at the center of a uniformly charged half-circle. However, if q is moved slightly off-center, the problem cannot be solved by elementary methods of calculus. The numerical approach will work equally well for q both on- and off-center.

Don't think that the numerical approach is always applicable. Try evaluating

$$\frac{(1+x)^{1/2}-1}{x}$$

for $x = 10^{-30}$. To 30 decimal places, the answer is nearly one-half, but your calculator will give you zero because it doesn't keep numbers to 30 places. However, by making a straight line approximation to $(1+x)^{1/2}$ near $x = 0$, with slope at $x = 0$ given by $(d/dx)(1+x)^{1/2}|_{x=0} = \frac{1}{2}(1+x)^{-1/2}|_{x=0} = \frac{1}{2}$, we can obtain the desired result.

2.8 Study and Problem Solving Strategy

As you have seen, the subject of electricity and magnetism, or E&M to the cognoscenti, requires a mathematical background in algebra, geometry, and trigonometry. Chapter 1 required integral calculus, and the current chapter requires vectors and integral calculus.

2.8.1 *Some Advice on How to Succeed in E&M*

This chapter introduces more difficult material, involving both vectors and calculus. *In performing integrals over vectors, first obtain the small vector you are adding up, and then find its components.* Only after this should you consider the integral calculus aspect (which involves, after all, just a method to perform summation). Many students worry so much about getting the calculus right that they miss the vector aspects of a problem.

The biggest hindrance to student understanding is an inability to see common-sense simplicity. There is a natural and understandable reason for this; many students are so involved in learning how to perform technical details that they don't see the forest for the trees. If you can't efficiently and correctly deal with the details, then you won't have the leisure to sit back and think about the overview; you'll be exhausted simply by the task of getting the details right. Nevertheless, you have not completed a problem until you have looked back on it and asked the common sense questions that your grandmother might ask: for example, "are there any comparisons that can be made to related problems?" In the next chapter, we will study the electric field along the axis of a uniform disk of charge. Far away from the disk, the field should look like that for a point charge; up close it should look like that for an infinite sheet. Both of these extreme cases provide common sense checks, and are questions that could be asked by someone who has not even studied E&M!

Finally, a word about *proportionality and scaling*. In Section 2.4, we determined the force between two protons in a helium nucleus not by computing it directly, but by comparing it with the already known force between a proton and an electron in a hydrogen atom. We used the fact that the magnitudes of the charges on the electron and proton are the same, but that the separation of two protons in the nucleus is a factor of 10^{-5} smaller than the electron–proton separation in the atom. Then, applying the inverse square law for electricity, we deduced that the Coulomb repulsion between two protons in the nucleus is 10^{10} bigger than the Coulomb attraction between an electron and proton in a hydrogen atom. This type of *proportional reasoning* is essential in scientific and engineering problems. Scientists and engineers repeatedly must consider how certain effects scale as various dimensions or velocities change. They don't recompute or remeasure everything—when appropriate, they scale the results. This is the basis of wind tunnels, for example, as used in aircraft design. Successful science and engineering students know how to employ this method of reasoning.

2.8.2 *Some Comments about Problem Solving*

Asking your own questions. Problems do not come out of nowhere. Someone has to think them up. For this book, the author had to think them up—with the assistance of a vast array of problems available from other books on this subject. Here is a secret. Not only can problems be solved, *they can also be made up.* You can do it yourself (it is an example of what has been called *active learning*). For each chapter, spend a minute or two thinking about how to make up an interesting variant on at least one problem. Here are some possibilities.

1. If you don't know what an equation means, try putting in numbers or, if appropriate, try drawing a graph. This is the most important rule of all! It's how scientists and engineers get started when they confront an equation whose meaning they don't understand.
2. Think about how to turn a doable problem to an undoable one.
3. Think about how to turn an undoable problem to a doable one—and do it.
4. Think about how to design an experiment. For example, in the electrometer problem, the string might have a certain breaking strength, and we want to know how much charge we can put on an object before the tension exceeds this breaking strength.
5. Make up a problem in which numbers, graphs, and equations are relevant; in the problems on the cancellation of two electrical forces, a sketch of the strengths of each force as a function of position is very revealing.
6. Think up what-if questions.

Although the bread and butter of physics is its ability to give precise answers to difficult but well defined questions, you should avoid the tendency to think *exclusively* in terms of stylized, closed-form mathematics problems. Often a simple qualitative question, whose answer can be given with a simple yes or no, or a direction, or greater than or less than, can teach a concept more efficiently

For Those Who Don't Think Problem Solving Is Important

"You can't learn to swim if you don't jump in the water."

—Anonymous

"Tourist to passerby in Manhattan: How do you get to Carnegie Hall?"
Violinist Jascha Heifitz, without breaking stride: "Practice, practice, practice!"

"To learn to play the blues, first you have to learn to play one song really well."

—Mance Lipscomb (1971), musician from Navasota, Texas

"The secret to success is being able to find more than one way to get the job done."

—Anonymous

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and effectively than a full-blown problem that requires an enormous amount of calculation. In any situation, even outside physics, one must be careful not to lose the overview in a confusion of details.

Thinking clearly. When you begin a problem, draw a figure in which the variables are clearly defined. When you complete a problem, ask, "What have I learned?" Think about how to modify the problem, and ask, "What changes does the modification cause?" Learn how to recognize problems that you have seen before, even when in disguise: if an automobile mechanic knows how to change a tire on a Ford, he cannot not plead ignorance when someone brings him a Chevrolet.

Styles of studying. There are different styles of studying. Many students work by themselves and don't spend much time giving explanations to others. In so doing, they lose the opportunity to learn while explaining. On the other hand, those who work only in groups are missing the opportunity to build their intellectual muscles; you don't go to the gym to watch others get in shape. Everyone should be doing some of both.

Problems

2-2.1 (a) If, in Robison's experiment (see Figure 2.1), q_1 (with mass m_1) is directly above q_2, show that the equilibrium condition is $kq_1q_2/r^2 = m_1g$. Neglect the mass of the holder. (b) If m_1 doubles, how does r change? Does the dependence of r on m_1 make sense qualitatively? (c) For $q_1 = q_2 = q$, $m_1 = 75$ g, and $r = 4.5$ cm, find q. (d) How would the equilibrium condition change if the top arm (held at a loose pivot) had mass M and length l?

2-2.2 (a) In a Coulomb's law experiment, as in Figure 2.2, a torque of 1.73×10^{-4} N-m is measured for a 2° twist. Find the torsion constant κ. (b) $q_1 = q_2 = 2.4 \times 10^{-8}$ C causes a twist of 5°. Find r.

2-3.1 Joan and Laura are separated by 15 m. Joan has a charge of 4.50×10^{-8} C, and Laura has a charge of -2.65×10^{-6} C. Find the force between them, and indicate whether it is attractive or repulsive.

2-3.2 Two regions of a thundercloud have charges of ± 5 C. Treating them as point charges a distance 3 km apart, determine the force between these regions of charge, and indicate whether it is attractive or repulsive.

2-3.3 The basketball player Michael Jordan is about 2 m tall, and weighs about 90 kg. What equal charges would have to be placed at his feet and his head to produce an electrical repulsion of the same magnitude as his weight?

2-3.4 Two point charges are separated by 2.8 cm. The force between them is 8.4 mN, and the sum of their charges is zero. Find their individual charges, and indicate whether the force is attractive or repulsive.

2-3.5 Show that, at fixed separation a, the maximum repulsion between two point charges of total charge Q occurs when each charge equals $Q/2$.

2-3.6 Two point charges sum to $-5\ \mu$C. At a separation of 2 cm, they exert a force of 80 N on each other. Find the two charges for the cases when (a) the forces are attractive, and (b) when they are repulsive. [Answer: (a) $q_1 = -5.63\ \mu$C, $q_2 = 0.63\ \mu$C; (b) $q_1 = -4.14\ \mu$C, $q_2 = -0.858\ \mu$C.]

2-3.7 Consider two space ships of mass $M = 1000$ kg in outer space. What equal and opposite charges would have to be given to them so that once an earth day they make circular orbits about their center at a separation of 200 m? Neglect their gravitational interaction, which at that distance would cause them to orbit only every 563 days.

2-3.8 Two point charges sum to $-0.5\ \mu$C. At a separation of 2 cm, they exert a force of 80 N on each other. Find the two charges for the cases when (a) the forces are attractive, and (b) when they are repulsive.

2-3.9 In esu-cgs units, where length is measured in cm, time in s, and mass in g, we take $k_{esu} = 1$. (a) Find the esu unit of charge, called the *statcoulomb* (sC) in terms of the SI unit of charge (C). (b) Find the charge of an electron in esu units. [Answer: 1 C $= 3 \times 10^9$ sC, $e_{esu} = 4.8 \times 10^{-10}$ sC.]

2-4.1 Refer to the electrometer example of Figure 2.4. Let the tension at breaking be $T_{max} = 2mg$. Find the value of θ_{max} and an algebraic expression for q_{max}.

2-4.2 In the electrometer example, find the angles and tensions if $q_L = 0.925 \times 10^{-8}$ C and $q_R = 3.7 \times 10^{-8}$ C? (*Hint:* No complex calculation is necessary. You can use results already obtained. If you are stumped, have a look at Problem 2.4.4a.)

2-4.3 In the electrometer example of Figure 2.4, if the angles are θ for $q_L = q_R = q$, what are they for $q_L = 2q$, $q_R = \frac{1}{2}q$? *Hint:* If you are stumped, have a look at Problem 2-4.4(a).

2-4.4 Two identical masses are suspended by identical strings. If the charges are the same, the strings make equal angles to the vertical. (a) If the charges are different, are the angles different? (b) If the masses are different, are the angles different? (*Hint:* Make the charges or masses very different.)

2-4.5 A positively charged bead q can slide without friction around a vertical hoop of radius R. A fixed positive charge Q is at the bottom of the hoop. Counterclockwise angular displacements of q relative to Q correspond to $\theta > 0$. (a) Find the electrical force on q, as a function of θ. (b) Find the component of this force along the hoop. See Figure 2.12.

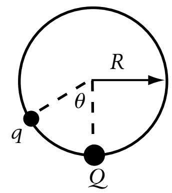

Figure 2.12 Problem 2-4.5.

2-5.1 Let $Q_1 = 5 \times 10^{-8}$ C be at $(0, 0)$, and $Q_2 = -4 \times 10^{-8}$ C be at $(3, 0)$, in m. A charge Q_3 is placed somewhere on the x-axis where the force on Q_3 is zero. (a) If the value of Q_3 is adjusted so that the force on Q_1 is zero, find the force on Q_2. (b) Find where Q_3 should be placed to feel no net force. (c) Find the value of Q_3 that will make Q_1 feel zero net force.

2-5.2 Consider two charges, $Q_1 = 5 \times 10^{-8}$ C at (1 cm, 0), and $Q_2 = -4 \times 10^{-8}$ C at (−2 cm, 4 cm). We want to find the position where a third charge Q should be placed for it to feel zero net force. (a) Present two methods for doing this. (b) Solve the problem by either method.

2-5.3 Three charges are at the corners of an equilateral triangle of side $l = 10$ cm. See Figure 2.13. If 2 μC is at the origin, and $-3\ \mu$C is at $(l, 0)$, find the force on 4 μC, at $(l/2, \sqrt{3}l/2)$.

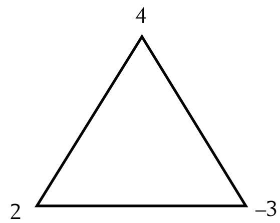

Figure 2.13 Problem 2-5.3.

2-5.4 Three charges Q are placed at the corners $(0, 0)$, $(a, 0)$, and $(0, a)$ of a square. See Figure 2.14. (a) Find the force on q placed at (a, a). (b) Find the force on q placed at $(a/2, -a/2)$. (c) Find the force on q placed at $(-a, -a)$.

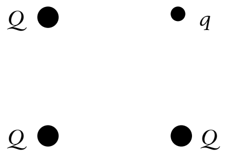

Figure 2.14 Problem 2-5.4.

2-5.5 Repeat the force calculation for the example of the addition of two noncollinear forces on a charge q. However, now let q_1 be at the origin, and rotate about the z-axis to make q_2 along the new x-axis. See Figure 2.15. To produce this requires, in Figure 2.7, rotating the line connecting q_1 and q_2 by an angle ϕ, where $m = \tan\phi = (r_2 \sin\theta_2 - 0)/((r_2 \cos\theta_2 - r_1) = -8.800$. Thus, in radian measure, $\phi = -83.52°(\pi/180) + \pi = 1.683$ radians. (We add π because we know, by Figure 2.7, that ϕ is in the first or second quadrant, whereas the calculator returns only a value in the first or fourth quadrant.) In degrees, $\phi = 96.5°$. By putting q_1 at the origin, and rotating clockwise by ϕ, we put q_2 along the x-axis, at a distance $r_{12} = \sqrt{(r_2 \sin\theta_2 - 0)^2 + ((r_2 \cos\theta_2 - r_1)^2} = 0.247m$, and we put q a distance r_1 away from the origin, at an angle of $\pi - \phi = 1.458$ radians, or 83.52° to the x-axis. Verify that $|\vec{F}|$ is the same as before, and that it is rotated clockwise by 96.5° relative to its previous value. This property, that the force rotates by the same angle as the coordinate system, is what we mean when we say that force is a vector. Thus Figure 2.15 was obtained by rotating Figure 2.7.

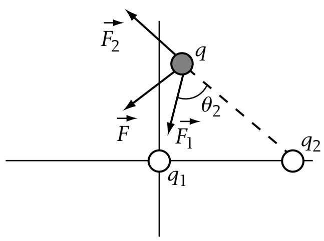

Figure 2.15 Problem 2-5.5.

2-6.1 Use the general symmetry argument to show that, along the perpendicular bisector of the uniformly charged rod discussed in Section 2.7, $F_y = 0$. *Hint:* Assume that $F_y \neq 0$, and then consider how the rod and F_y transform under rotations of 180° about the x-axis. Would the argument work if you considered a reflection $(x, y, z) \to (x, -y, z)$ of the rod and F_y?

2-6.2 An octagon has charges $-q$ at each of its eight vertices, and charge Q at its center. See Figure 2.16. (a) Use a general symmetry argument to show that the force on Q is zero. (b) Let the charges $-q$ be replaced by infinitely long line charges $-\lambda$ normal to the page. Use a general symmetry argument to show that the force on Q is zero. (To prove this, we don't even need to know the force between Q and a line charge!)

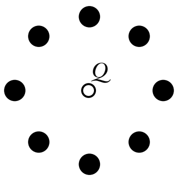

Figure 2.16 Problem 2-6.2.

2-7.1 A line charge with total charge $Q > 0$ uniformly distributed over its length l extends from $(0, 0)$ to $(l, 0)$. See Figure 2.17. (a) Find the force on a charge $q > 0$ placed a distance a to its right, at $(l + a, 0)$. (b) Verify that this force has the expected inverse square behavior for large a.

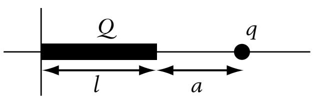

Figure 2.17 Problem 2-7.1.

2-7.2 A long rod of charge per unit length $\lambda > 0$ is normal to the xy-plane and passes through the origin. In addition, a charge $Q > 0$ is located at $(0, a, 0)$. See Figure 2.18. Find the position where a third charge q will feel zero force.

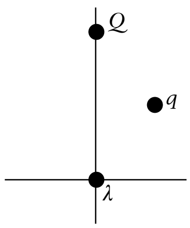

Figure 2.18 Problem 2-7.2.

2-7.3 Use the computational symmetry argument to show that $F_y = 0$ along the perpendicular bisector of the uniformly charged rod of Section 2.7.

2-7.4 A line charge with total charge $Q > 0$ uniformly distributed over its length l extends from $(0, 0)$ to $(l, 0)$. Find the force on a charge q that is placed anywhere in the xy-plane.

2-7.5 Find the force on a charge q at (a, a) due to a charge Q uniformly distributed over a rod of length L with one end at $(0, 0)$ and the other end at $(0, L)$. See Figure 2.19.

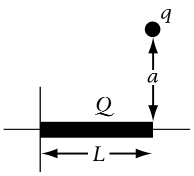

Figure 2.19 Problem 2-7.5.

2-7.6 A rod of length a whose ends are at $(0, 0)$ and $(a, 0)$ has a charge density $\lambda = (Q_0/a^2)x$. See Figure 2.20. (a) Find the total charge Q on the rod. (b) Find the force on a charge q at $(-b, 0)$. (c) Verify that the force has the correct limit as $b \to \infty$.

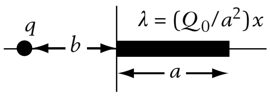

Figure 2.20 Problem 2-7.6.

2-7.7 A charge Q is uniformly distributed over the upper half of a circle of radius a, centered at the origin. See Figure 2.21. Find the force on a charge q at the origin.

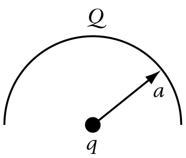

Figure 2.21 Problem 2-7.7.

2-7.8 A charge Q is uniformly distributed over the first quadrant of a circle of radius a. See Figure 2.22. Find the force on a charge q at the center of the semicircle.

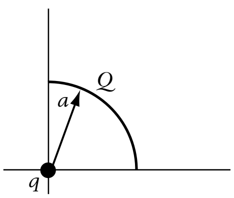

Figure 2.22 Problem 2-7.8.

2-7.9 A charge Q is uniformly distributed over an arc of radius a and angle α that extends from the x-axis counterclockwise. See Figure 2.23. Find the force on a charge q at the center of the semicircle.

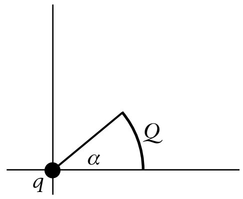

Figure 2.23 Problem 2-7.9.

2-G.1 Many electroscopes have a circular conducting base that is both above and connected to a charge detector device (often a needle that can rotate relative to its mount, or flexible foil). They work by a combination of electrostatic induction at the base (due to the source charge), and repulsion of like charges at the detector (needle or foil). See Figure 2.24. (a) If positive charge is brought near the base of an electroscope, what is the sign of the charge attracted to the base? (b) What kind of charge must be at the needle or foil?

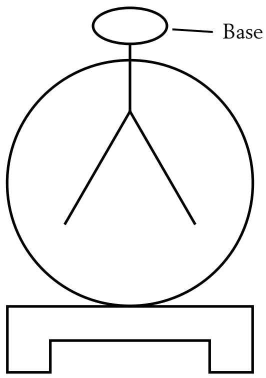

Figure 2.24 Problem 2-G.1.

2-G.2 Show that, for $|nx| \ll 1$, $(1 + x)^n \approx 1 + nx$. This gives the first two terms, for small x, in what is known as the MacLaurin expansion.

2-G.3 Determine the behavior of $(x^2 + 1)^{\frac{1}{2}} - x$ for large x; it approaches zero, but *how* does it approach zero?

2-G.4 In the spreadsheet calculation of Table 2.2, the sums for odd n seem to provide an upper limit for the integral. (a) Explain why. The sums for $n = 2, 4, 6, 8, 10$ seem to be a lower limit for the integral. However, for $n = 12$ and larger, the approximate value is *less* than the exact integral. (b) Explain why. *Hint:* For odd n, all the charge is approximated by charge that is nearer than it really is, but that is not the case for even n.

2-G.5 Two charges $-Q$ are fixed at $(0, a)$ and $(0, -a)$. A third charge, q, is constrained to move along the x-axis. See Figure 2.25. (a) Find the force on q for any value of $|x| < a$; (b) convince yourself that, for $|x| \ll a$, this force is just like that for a harmonic oscillator, and obtain the effective spring constant K; (c) if q has mass m, find the frequency of oscillation of q about the origin. *Hint:* Expand $(a \pm x)^{-2}$ for small x.

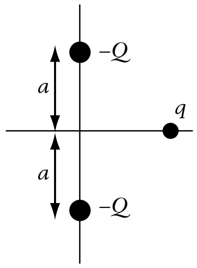

Figure 2.25 Problem 2-G.5.

2-G.6 In the previous problem, the motion of q was stable near the origin for motion along x. Discuss the stability of the motion if small displacements along the y-axis are allowed. Consider $x = 0$ and $|y| < a$.

2-G.7 A long rod of charge per unit length λ is held vertical. At its midpoint, one end of a massless string of finite length l is attached. At the other end of the string is a charge q, with mass m. See Figure 2.26. Find the equilibrium angle of the string to the vertical, and find the tension in the string at that angle.

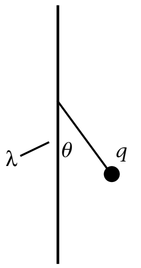

Figure 2.26 Problem 2-G.7.

2-G.8 A square of side a, of uniform charge per unit area σ, is centered about the origin of the xy-plane with sides parallel to the x- and y-axes. A charge q lies a distance l along its perpendicular bisector. See Figure 2.27. (a) By building up the square from lines in the xy-plane of area $l dz$, show that the force on q satisfies $|\vec{F}| = 4kq\sigma \sin^{-1}[a^2/(a^2 + 4l^2)]$. (b) Show that, as $l/a \to \infty$, $|\vec{F}| \to (kQq/l^2)$. (c) Show that, as $l/a \to 0$, $|\vec{F}| \to (2\pi k\sigma)q$. *Hint:* Modify (3.21) to $|d\vec{F}| = kqdQ/l\sqrt{l^2 + a^2/4}$, use the direction cosine factor $l\sqrt{l^2 + a^2/4}$, and then integrate over dF_x to get F_x. A trig substitution like $z = a \tan\theta$ may be helpful.

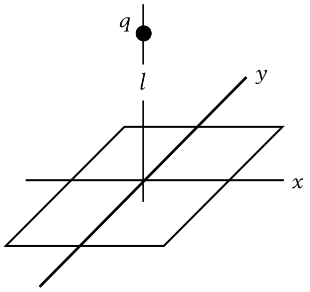

Figure 2.27 Problem 2-G.8.

2-G.9 Consider a ring of radius a, centered at the origin of the xy-plane. It is of uniform charge density and has total charge Q. A charge q lies on the x-axis a distance l from the origin. See Figure 2.28. (a) Find the force on q due to the circle, for $l > a$. (b) Find the force on q due to the ring, for $l < a$. This example shows that the charge on the ring does not behave as if it were centered at its geometrical center (except in the limit where $l/a \to \infty$).

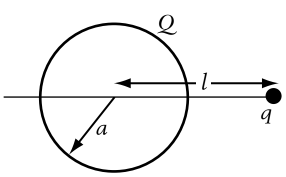

Figure 2.28 Problem 2-G.9.

2-G.10 Conducting spheres are subject to the amber effect. Hence, as two equally charged conducting spheres of radius a approach each other, in addition to the inverse square law of repulsion, there should be an effect due to the charge on one polarizing the other, and vice versa (i.e., electrostatic induction). Using advanced methods, it is possible to determine this effect exactly, but we already know enough to determine the most important contribution. When the separation r is only a few times a, this polarization effect can cause deviations from pure inverse square on the order of a few percent. See Figure 2.29. (a) Does polarization make the net force appear weaker or stronger? (b) What dependence on distance do you expect the most important correction to take? (c) How would you determine, from experimental data of force F versus separation r, how large a coefficient it has? *Hint*: For large r, plot $r^2 F$ *vs.* r^{-3}.

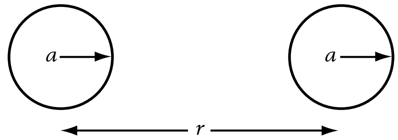

Figure 2.29 Problem 2-G.10.

2-G.11 Repeat the considerations given in Problem 2.6.10 for two conducting spheres of equal and opposite charge.